EE 120 Cheatsheet MT 2

Unit step & Kronecker Delta.

$$\boxed{u(n) = \sum_{k=0}^{\infty} \delta(n-k)}$$

let $m \triangleq n-k$

$$\boxed{u(n) = \sum_{m=n}^{-\infty} \delta(m)}$$

$$\boxed{\delta(n) = u(n) - u(n-1)}$$

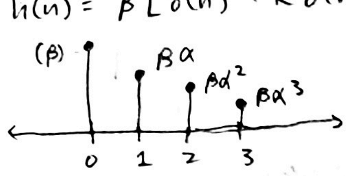
// discrete time derivative of sorts.

2-Day moving avg: $\frac{x(n) + x(n-1)}{2} = y(n)$

exponential moving avg: $\boxed{y(n) = \beta \sum_{k=0}^{\infty} \alpha^k x(n-k)}$

$h(n) = \beta[\delta(n) + k\delta(n-1) + k^2\delta(n-2) \ldots]$

$0 < \alpha < 1$



CONVOLUTION

$(\delta * g)(n) = g(n)$ // δ is the identity for convolution, g is impulse resp.

$(f*g)(n) = (g*f)(n)$ // commutativity of conv.

$(f*g)(n) = \delta(n)$ // inverse systems.

$$\boxed{\begin{aligned}(f*g)(t) &\triangleq \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} g(\tau)\, f(t-\tau)\, d\tau\end{aligned}}$$

$$\boxed{\begin{aligned}(f*g)(n) &= \sum_{k=-\infty}^{\infty} f(k)\, g(n-k) \\ &= \sum_{k=-\infty}^{\infty} g(k)\, f(n-k)\end{aligned}}$$

Euler's ID: $\boxed{e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)}$

Frequency Resp

1. let $x = e^{i\omega n}$ $\quad y = H(\omega)e^{i\omega n}$

   use algebra solve for $H(\omega)$
2. $\boxed{H(\omega) = \sum_{\text{all } n} h(n) e^{-i\omega n}}$

   NOT CONVOLUTION, JUST A DOT-PRODUCT.

Low-Pass filter form: $\frac{1+e^{-i\omega}}{2}$, $\frac{1}{1-\alpha e^{-i\omega}}$

More Euler's ID

$$\boxed{\cos(\omega t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2}}$$

$$\boxed{\sin(\omega t) = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}}$$

Test	Method
Time-Invariance	$\boxed{\text{let } \hat{x}(n) = x(n-N) \text{ Does } \hat{y}(n) = y(n-N)\,?}$
Linear	let $\boxed{\hat{x}(n) = a_1x_1 + a_2x_2}$ does $\boxed{\hat{y}(n) = a_1y_1 + a_2y_2\,?}$ ① Scaling ② homogenaity.

Periodicity at what N does $e^{in} = e^{i(n+N)}$

$N = 2\pi k$ where $k \in \mathbb{Z}$

(in discrete case, if N is never an integer, system is NOT periodic)

at what T does $x(t) = x(t+T)$?

lowest val = fund. period.

Causality: must have a FULLY characterized system to prove causality.

BUT, can use simple counterexamples to disprove causality.

LTI: causal H iff $\boxed{h(n) = 0 \ \forall n < 0}$ $\boxed{h(t) = 0 \ \forall t < 0}$

"impulse resp must be 0 when time/sample is negative"

Non-LTI: more rigorous, creative method of proving "prediction" is necessary.

BIBO Stability LTI: BIBO $\Leftrightarrow$ $\boxed{\sum_{\text{all } n} |h(n)| < \infty \quad \int_{-\infty}^{\infty} |h(t)|\,dt < \infty}$ absolute summability

Non LTI: Find bounded input $x < B_x$ where all output $y < B_y$ ($B_x \neq B_y$ necessarily)

LCCDE

Discrete: $F(\omega) \leftrightarrow f(n)$ imp. resp.

$$\boxed{F(\omega) = \tfrac{1}{3}e^{i\omega} + \tfrac{1}{3} + \tfrac{1}{3}e^{-i\omega}}$$

comfort.

$$\begin{aligned} y(n) &= [\tfrac{1}{3}e^{i\omega} + \tfrac{1}{3} + \tfrac{1}{3}e^{-i\omega}]\,x(n) \\ &= \tfrac{1}{3}x(n+1) + \tfrac{1}{3}x(n) + \tfrac{1}{3}x(n-1)\end{aligned}$$

$$\boxed{f(n) = \tfrac{1}{3}\delta(n+1) + \tfrac{1}{3}\delta(n) + \tfrac{1}{3}\delta(n-1)}$$

State-Space Discrete:

$$\begin{bmatrix} q_1(n+1) \\ q_2(n+1)\end{bmatrix} = [\underline{A}]\begin{bmatrix} q_1(n) \\ q_2(n)\end{bmatrix} + [\vec{B}]\,x$$

set $q_1 = y(n-1)$

$q_2 = y(n-2)$

$q_2(n+1) = q_1$!

DIRAC DELTA

$\int_{-\infty}^{\infty} x(t)\,\delta_n(t)\,dt = \lim_{n\to\infty} \frac{n}{2}\int_{-1/n}^{1/n} x(t)\,dt.$ AREA Under Dirac delta $= 1$!

SIFTING: $\boxed{\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)}$

SAMPLING: $\boxed{x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)}$

$$\boxed{u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau \begin{cases} 0 & t < 0 \\ 1 & t \geq 0\end{cases}}$$

unit step CT.

$$\boxed{\dot{u}(t) = \delta(t)}$$

IRL systems

Intro: CT exponential.

let $\boxed{g(t) = \beta e^{-\alpha t} u(t)}$

$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt$

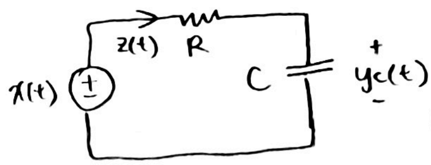
$= \beta \int_{-\infty}^{\infty} e^{(-\alpha - i\omega)t} u(t) dt$

$= \beta \int_{0}^{\infty} e^{(-\alpha - i\omega)t} dt$

$= \boxed{\beta / \alpha + i\omega}$

COMFORT.

R-C circuit.



LCCDE:

$\dot{y_c}(t) = \frac{1}{RC}(x(t) - y_c(t))$

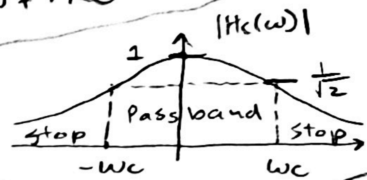
$\boxed{y_c(t) = RC\dot{y_c}(t) - x(t)}$

$\boxed{H_c(\omega) = \frac{1}{RCi\omega + 1}} = \frac{1/RC}{i\omega + 1/RC}$

// follows

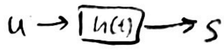
$\boxed{h(t) = \frac{1}{RC} e^{-t/RC} u(t)}$

$\boxed{\omega_c = 1/RC}$



LOW-PASS! CAUSAL ✓ BIBO Stable ✓

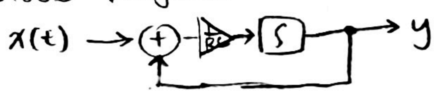
Aside: CT step resp



$s = \int_{-\infty}^{\infty} h(\tau) u(t - \tau) d\tau$ "convolve"

$\boxed{s = \int_{-\infty}^{t} h(\tau) d\tau}$ zeros out everywhere else.

Block Diagram:



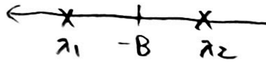
LCCDE:

$\boxed{H(\omega) = \frac{\Sigma b_m (i\omega)^m}{\Sigma a_n (i\omega)^n}}$

// think of $(i\omega)$ as a derivative of sorts.

MODES OF SYSTEM:

① Two <u>real</u> negative eigenvals

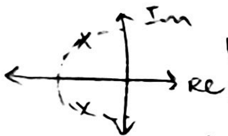


OVERDAMPED
NO OSCILLATION

② Two IDENTICAL eig vals.

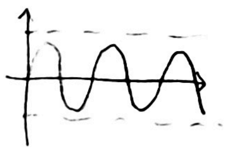
CRIT. DAMPED
NO OCILLAT.

③ Two COMPLEX CONJ. eigvals.

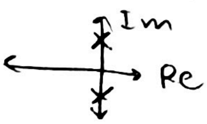


UNDERDAMPED
OSCILLATIONS
still decays

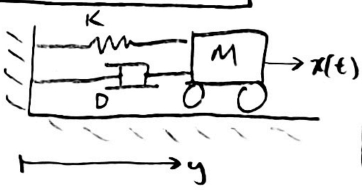
④ Un-damped. Real component of eigvals = 0.



PURE OSCILLAT



MASS-SPRING



$\boxed{M\ddot{y} = -D\dot{y} - Ky + x(t)}$

$\boxed{H(\omega) = \frac{1}{M(i\omega)^2 + Di\omega + K}}$

SS let $q_1 = y$ $q_2 = \dot{y}$

$$\begin{bmatrix} \dot{q_1} \\ \dot{q_2} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{D}{M} \end{bmatrix}}_{A} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1/M \end{bmatrix}}_{\vec{B}} x$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \underbrace{0}_{D} x(t)$$

$\dot{q}(t) = Aq(t)$

solution:

$e^{At}\vec{q}(0) = \vec{q}(t)$

$$\boxed{e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = I + At + \frac{(At)^2}{2!} + \ldots}$$

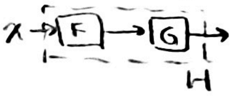
$\boxed{A = V\Lambda V^{-1}}$ $\boxed{\vec{q}(0) = c_1\vec{u_1} + c_2\vec{u_2}}$

//spectral theorem for lin ind. eigenvectors.

$e^{At}\vec{q}(0) = c_1 e^{At}\vec{u_1} + c_2 e^{At}\vec{u_2}$

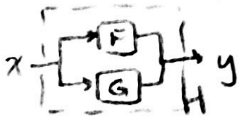
$= \boxed{c_1 e^{\lambda_1 t}\vec{u_1} + c_2 e^{\lambda_2 t}\vec{u_2}}$ ☆

Connected systems



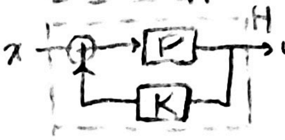
$H(\omega) = F(\omega)G(\omega)$

$h(t) = (f * g)(t)$



$H(\omega) = F(\omega) + G(\omega)$

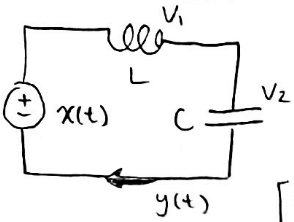
$h(t) = f(t) + g(t)$



$\boxed{H(\omega) = \frac{P}{1 - KP}}$

$\frac{\text{"Forward Gain"}}{1 - \text{"Loop Gain"}}$

L-C Circuit



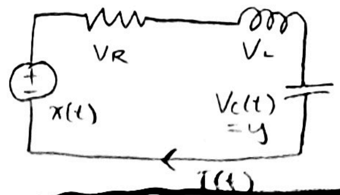
$\boxed{y = C\dot{x} - LC\ddot{y}}$

let $q_1 = y$
$q_2 = \dot{y}$

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & 0 \end{bmatrix}}_{A} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}}_{B} \dot{x}$$

$$\boxed{H(\omega) = \frac{Ci\omega}{1 + LC(i\omega)^2}}$$

RLC Circuit



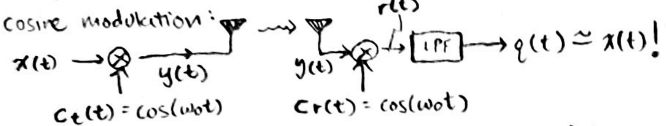
$$\boxed{C\ddot{y} = \frac{-y}{L} - \frac{\dot{y}RC}{L} + \frac{x}{L}}$$

$$\begin{bmatrix} \dot{I} \\ \dot{V}_c \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} I \\ V_c \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} x$$

$\boxed{\Omega = \omega T_s}$ $\boxed{\omega_s = \frac{2\pi}{T_s}}$! $\boxed{\omega = \frac{\Omega}{T_s}}$

AMPLITUDE MODULATION

cosine modulation:

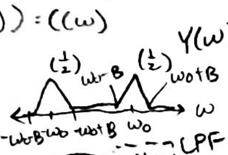


What is involved

$x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$

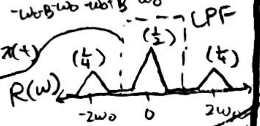
$c(t) = \cos(\omega_0 t) \xleftrightarrow[CTFT]{\mathcal{F}} (\delta(\omega+\omega_0) + \delta(\omega-\omega_0)) : C(\omega)$

$y(t) = x(t)c(t) \xleftrightarrow{\mathcal{F}} Y(\omega) = \frac{1}{2\pi}(X \star C)(\omega)$



$r(t) = y(t)c(t)$
$= \cos^2(\omega_0 t)\, x(t) = \frac{1}{2}[1 + \cos(\omega_0 t)]x(t)$

$R(\omega) = \frac{1}{2}X(\omega) +$ Boxed off $= Q(\omega)$
$\simeq X(\omega)$!

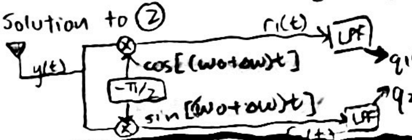


FOR NO OVERLAP: $\boxed{\omega_0 \geq B}$ ← why the difference

Problems ① out-of-phase oscillators.
$c(t) = \cos(\omega_0 t + \theta) \rightarrow r(t) = \cos(\omega_0 t)\cos(\omega_0 t + \theta)x(t)$
$= \frac{1}{2}x(t)[\cos(2\omega_0 t + \theta) + \cos(\theta)]$
phase err; LPF kills it; scaling error

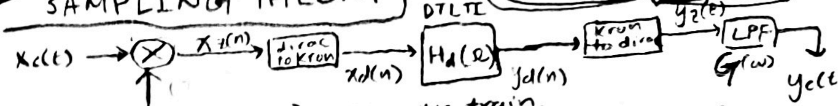
② frequency drift: $c(t) = \cos((\omega_0 + \Delta\omega)t)$
drift comp.
$r(t) = \frac{1}{2}x(t)[\cos((2\omega_0 + \Delta\omega)t) + \cos(\Delta\omega t)]$

Solution to ②



$$\boxed{x(t) = \sqrt{q_1(t)^2 + q_2(t)^2}}$$
$q_1(t) = \cos(\Delta\omega t)x(t)$
$q_2(t) = \sin(\Delta\omega t)x(t)$

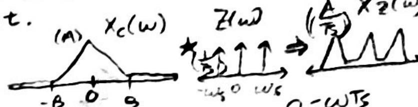
SAMPLING THEORY



$z(t) = \sum_{\ell} (\delta(t - \ell T_s)) \rightarrow$ impulse train.

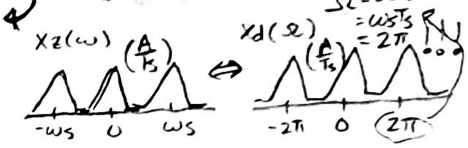
$x_z(t) = x_c(t)z(t) \xleftrightarrow{\mathcal{F}} X_z(\omega) = \frac{1}{2\pi}(X_c \star Z)(\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)$

$z(t) \xleftrightarrow{\mathcal{F}} Z(\omega) = \frac{1}{T_s}\sum_k e^{ik\omega_s t}$.



$X_d(\Omega)|_{\Omega = \omega T_s} = X_z(\omega)$

$X_z(\omega)|_{\omega = \frac{\Omega}{T_s}} = X_d(\Omega)$



FOR NO ALIASING: $\omega_s \geq 2B$

$y_d(n) = x_d(n) \star h_d(n)$

$Y_d(\Omega) = H_d(\Omega)X_d(\Omega) = H_d(\Omega)X_z(\omega)|_{\omega = \frac{\Omega}{T_s}}$
$= H_d(\Omega)X_c(\omega)|_{\omega = \frac{\Omega}{T_s}} \star \frac{1}{T_s}$

$Y_c(\omega) = G(\omega)Y_z(\omega)$
$Y_z(\omega) = Y_d(\Omega)|_{\Omega = \omega T_r}$

$\Rightarrow Y_c(\omega) = G(\omega)Y_d(\Omega)|_{\Omega = \omega T_r}$

$$\boxed{Y_c(\omega) = \frac{G(\omega)}{T_s}H_d(\omega T_r)X_c\left(\frac{\omega T_r}{T_s}\right)}$$

Extra Z-transform from discussion

Unilateral Z-transform
$\hat{X}(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$
//one-sided.

$y(n) = y_{ZIR}(n) + y_{ZSR}(n)$

ZIR: zero-input resp $y(-1) \neq 0$, $x(n) = 0$
ZSR: zero state resp $y(-1) = 0$, $x(n) \neq 0$

$$\hat{X}_{RoC} \cap \hat{H}_{RoC} \in \hat{Y}_{RoC}$$
intersect of X_{RoC} & H_{RoC} must be in $\hat{Y}_{RoC}$

Z-transform

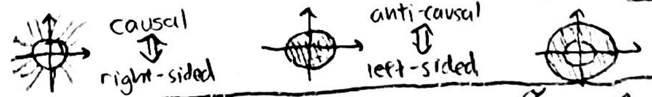
origins: let $x(n) = z^n$ where $z \in \mathbb{C}$ $Re^{i\omega t}$

$x(n) = z^n \rightarrow \boxed{h(n)} \rightarrow y(n) = \sum_m h(m)x(n-m)$
$= \sum_m h(m)z^{n-m}$
$= \underbrace{\sum_m h(m)z^{-m}}_{\hat{H}(z)} z^n$

$$\boxed{\hat{H}(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n}}$$

If RoC includes Unit circle: STABLE; $\mathcal{Z} \longleftrightarrow \mathcal{F}$
$X(\omega) = \hat{X}(z)|_{z = e^{i\omega}}$

If RoC is OUTSIDE of OUTERMOST POLE, causal.



convolution:	$y(n) = (x \star h)(n) \xrightarrow{\mathcal{Z}} \hat{Y}(z) = \hat{X}(z)\hat{H}(z)$
time shift:	$x(n-1) \xleftrightarrow{\mathcal{Z}} \hat{X}(z)z^{-1}$ OR $x(-n-1) \xleftrightarrow{\mathcal{Z}} \hat{X}(z)z^{1}$
LCCDEs:	$\hat{H}(z) = \hat{Y}(z)/\hat{X}(z) = \sum_{m=0}^{M} b_m z^{-m} / \sum_{k=0}^{\hat{N}} a_k z^{-k}$ $a_k \rightarrow x$ coeff, $b_m \rightarrow y$ coeff.

Initial-Value Thm: $x(0) = \lim_{n\to 0} x(n) = \lim_{z\to\infty} \hat{X}(z)$

Common z-transforms

	RoC
$\delta[n] \xleftrightarrow{\mathcal{Z}} 1$	All z
$u[n] \xleftrightarrow{\mathcal{Z}} \frac{1}{1-z^{-1}} = \frac{z}{z+1}$	$\lvert z\rvert > 1$
$-u[n-1] \xleftrightarrow{\mathcal{Z}} \frac{1}{1-z^{-1}} = \frac{z}{z+1}$	$\lvert z\rvert < 1$
$\delta[n-m] \xleftrightarrow{\mathcal{Z}} z^{-m}$	All z except $0, \infty$
$a^n u[n] \xleftrightarrow{\mathcal{Z}} \frac{1}{1-az^{-1}}$	$\lvert z\rvert > a$
$-a^n u[-n-1] \xleftrightarrow{\mathcal{Z}} \frac{1}{1-az^{-1}}$	$\lvert z\rvert < a$

closely related

Graph-matching:

① put all $\hat{H}(z)$ in terms of z^{+m} → $\boxed{\frac{1}{1-az^{-1}}}$
② if pole is $|z| < 1$ & ⊕ → decay; ⊖ → oscillatory decay → $\boxed{\frac{1}{1+az^{-1}}}$
③ # of poles $\xrightarrow{|H| \text{ plot}}$ # and loc of spikes
# of zeros → # and loc of local max/minima

DTFT.

$$x(n) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(\omega)e^{i\omega n}\,d\omega$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-i\omega n}$$

CTFT

$$x(t) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(\omega)e^{i\omega t}\,d\omega$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}\,dt$$

intuitivites:

$$x(-n/-t) \xleftrightarrow{\mathcal{F}} X(-\omega)$$

$$x(n-n_0) \xleftrightarrow{\mathcal{F}} X(\omega)e^{-i\omega n_0}$$

$$x(t-t_0) \xleftrightarrow{\mathcal{F}} X(\omega)e^{-i\omega t_0}$$

$$x(n) \star y(n) \xleftrightarrow{\mathcal{F}} X(\omega)Y(\omega)$$

$$x(n)y(n) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi}(X \circledast Y)(\omega)$$

circular convolution: convolve ONLY 1 period of X over Y

$$x(t) \star y(t) \xleftrightarrow{\mathcal{F}} X(\omega)Y(\omega)$$

$$x(t)y(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi}(X \star Y)(\omega)$$ → IMPORTANT for amplitude modulation

$$x^*(n) \xleftrightarrow{\mathcal{F}} X^*(-\omega)$$

$$x^*(t) \xleftrightarrow{\mathcal{F}} X^*(-t)$$ → it follows then... if $x(n) \in Re$,

$$x^*(n) = x(n) \xleftrightarrow{\mathcal{F}} X^*(-\omega) = X(\omega)$$

$$\Rightarrow X(-\omega) = X^*(\omega)$$

Exponentials.

$$e^{i\omega_0 n} \xleftrightarrow{\mathcal{F}} 2\pi\delta(\omega-\omega_0)$$

$$\delta(n-n_0) \xleftrightarrow{\mathcal{F}} e^{-i\omega n_0}$$

$$e^{i\omega_0 t} \xleftrightarrow{\mathcal{F}} 2\pi\delta(\omega-\omega_0)$$

$$\delta(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-i\omega t_0}$$

even/odd.

$$Re\{X(\omega)\} \xleftrightarrow{\mathcal{F}} x_{even}(n)$$

$$Im\{X(\omega)\} \xleftrightarrow{\mathcal{F}} x_{odd}(n)$$

$$x_{even}(n) + x_{odd}(n) = x(n)$$

Duality

$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega) \Rightarrow X(t) \xleftrightarrow{\mathcal{F}} 2\pi\, x(-\omega)$$

Inner Product Def.

$$\langle F, G\rangle \triangleq \int_{\langle 2\pi\rangle} F(\omega)G^*(\omega)\,d\omega$$

$$\hookrightarrow \langle \phi_n, \phi_\ell\rangle = 2\pi\delta(n-\ell)$$

$$\langle F, G\rangle \triangleq \int_{\langle 2\pi\rangle} F(t)G^*(t)\,dt$$

$$\hookrightarrow \langle \phi_t, \phi_\ell\rangle = 2\pi\delta(t-\ell)$$

DTFS

$$x(n) = \sum_{k=\langle p\rangle} X_k e^{ik\omega_0 n}$$

$$X_k = \frac{1}{p}\sum_{n=\langle p\rangle} x(n)e^{-ik\omega_0 n}$$

CTFS

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t}$$

$$X_k = \frac{1}{p}\int_{\langle p\rangle} x(t)e^{-ik\omega_0 t}\,dt.$$

where $\psi_k(t) = e^{i\omega_0 k t}$.

$$\langle f, g\rangle \triangleq \sum_{\langle p\rangle} f\, g^*$$

$$\hookrightarrow \langle \psi_k, \psi_\ell\rangle = P\delta(k-\ell)$$

$$\langle f, g\rangle \triangleq \int_{\langle p\rangle} f\cdot g^*\,dt.$$

$$\hookrightarrow \langle \psi_k, \psi_\ell\rangle = P\delta(k-\ell)$$

JUST MATH:

$$\cos(\omega_0 t) = \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}$$

$$\sin(\omega_0 t) = \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i}$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$$

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$

$$\cos^2\theta = \frac{1-\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1+\cos 2\theta}{2}$$

finite limit! not infinite

$$\boxed{\sum_{k=0}^{N} p^k = \frac{1-p^{N+1}}{1-p}}$$

$$\int_{\langle p\rangle}\cos(\omega)\,d\omega = \int_{\langle p\rangle}\sin(\omega)\,d\omega = \int e^{i\omega t}\,dt = 0$$

$$\sum_{n=0}^{N-1} e^{-i\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1}\left(e^{-2\pi i}\right)^{\frac{kn}{N}} = \sum_{n=0}^{N-1}(1)^{\frac{kn}{N}} = \boxed{N}$$