ME C134 final cheatsheet

Laplace Transforms

$$\mathcal{L}[f(t)] = \int_{0^-}^{\infty} f(t)e^{-st}\,dt = F(s)$$

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi}\int_{\sigma-j\omega}^{\sigma+j\omega} F(s)e^{st}\,dt = f(t)u(t)$$

Intuition: convolute e^{-st} across your input time-domain signal. where $s = \alpha + i\omega$

so you are convoluting e^{-at} & $e^{-i\omega t}$
exponentials sinusoids

LAPLACE T tells us which SINUSOIDS & EXPONENTIALS are most prevalent in your signal.

Common Laplace Transforms

f(t)	F(s)	
$\delta(t)$	1	// impulse function
$u(t)$	$1/s$	// unit-step
$t\,u(t)$	$1/s^2$	// ramp
$t^n u(t)$	$n!/s^{n+1}$	
$\sin\omega t\ u(t)$	$\omega/(s^2+\omega^2)$	
$\cos\omega t\ u(t)$	$s/(s^2+\omega^2)$	

Properties of Laplace Transforms

1. $\mathcal{L}[kf(t)] = k\mathcal{L}[f(t)]$; $\mathcal{L}[f(t)+g(t)] = \mathcal{L}[f(t)]+\mathcal{L}[g(t)]$
2. $\mathcal{L}[e^{-at}f(t)] = F(s+a)$
3. $\mathcal{L}[f(t-T)] = e^{-sT}F(s)$
4. $\mathcal{L}[f(at)] = \frac{1}{a}F(\frac{s}{a})$
5. $\mathcal{L}[df/dt] = sF(s) - f(0^-)$
6. $\mathcal{L}[\int_0^t f(\tau)d\tau] = F(s)/s$

IVT & FVT for $\mathcal{L}$

$f(\infty) = \lim_{s\to 0} sF(s)$ //FVT

$f(0^+) = \lim_{s\to\infty} sF(s)$ //IVT

Electro Mechanical Analogs

			Impedance
EE) Resistor:	$v(t) = iR$	$\overset{\mathcal{L}}{\Longleftrightarrow}$ $V(s) = RI(s)$	R
Capacitor:	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$\overset{\mathcal{L}}{\Longleftrightarrow}$ $V(s) = \frac{1}{Cs}I(s)$	$1/Cs$
inductor:	$v(t) = \frac{1}{L}\frac{di}{dt}$	$\overset{\mathcal{L}}{\Longleftrightarrow}$ $V(s) = sLI(s)$	Ls
ME) Spring:	$F(t) = kx(t)$	$\Longleftrightarrow$ $F(s) = kX(s)$	K
Dashpot:	$F(t) = f_v\,dx/dt$	$\Longleftrightarrow$ $F(s) = f_v sX(s)$	$f_v s$
Mass:	$F(t) = m\,d^2x/dt^2$	$\Longleftrightarrow$ $F(s) = ms^2X(s)$	ms^2

EE - ME conversion

KVL: $L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i(\tau)d\tau$

let $i(t) = C\frac{dV_c}{dt}$

$\Rightarrow LC\frac{d^2V_c}{dt^2} + RC\frac{dV_c}{dt} + V_c$

$\Updownarrow \mathcal{L}$

$\underbrace{LCs^2V_c(s)}_{\text{counterforce (M mass)}} + \underbrace{RCsV_c(s)}_{\text{damper } (f_v \text{ dashpot})} + \underbrace{V_c(s)}_{\text{driver (K spring)}}$

Nonlinearities (we MUST have a pivot point x_0 to linearize)

$$f(x) = f(x_0) + \frac{df(x_0)}{dt}(x-x_0) + \frac{d^2f(x_0)}{dt^2}\frac{(x-x_0)^2}{2!} + \dots + \frac{d^nf(x_0)}{dt^n}\frac{(x-x_0)^n}{n!}$$ //Taylor expansion

For us, 1st order apprx is enough: $\boxed{f(x) \approx f(x_0) + \frac{df(x_0)}{dt}(x-x_0)}$

STATE-SPACE representation

$\dot{x}(t) = Ax + Bu$

$y(t) = Cx + Du$

$$\boxed{\text{T.F.} \quad \frac{Y(s)}{U(s)} = C[(sI-A)^{-1}B] + D}$$

Controllable-canonical form

$$\text{T.F.} = \frac{b_2s^2 + b_1s + b_0}{s^3 + a_2s^2 + a_1s + a_0}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = [b_2\ b_1\ b_0] \quad D = [0]$$

always identity

negative denominator coeffs

positive numerator coeffs

TIME RESPONSE

1st-order no zeros

$G(s) = \frac{b}{s+a}$

$C(s) = R(s)G(s) = \frac{1}{s}\left(\frac{b}{s+a}\right)$

$\mathcal{L}^{-1}[C(s)] = (1 - e^{-at})u(t)$

FREE response due to system

FORCED response due to input.

Value	Expr.
Time const	$T_c = 1/a$
Rise Time	$T_r \simeq \frac{2.2}{a}$
Settling Time	$T_s = \pm 2\%$ ss value

General Form of 2nd-order

$$\boxed{G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}}$$

where
ω_n = "natural freq"
ζ = "damping ratio"

When

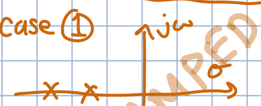
$\zeta = 0$	UNDAMPED
$\zeta = 1$	CRIT. DAMPED
$\zeta > 1$	OVERDAMPED
$0 < \zeta < 1$	UNDERDAMPED
$\zeta < 0$	UNSTABLE!

Poles

$$\boxed{P_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}}$$

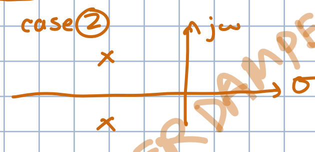
2ND order no zeros

case ① OVERDAMPED



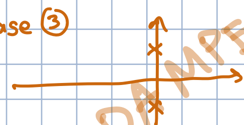
$c(t) = (1 + K_1e^{-\sigma_1 t} + K_2e^{-\sigma_2 t})u(t)$

case ② UNDERDAMPED

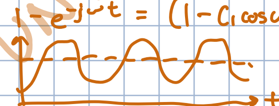


$c(t) = 1 - e^{-\sigma t}e^{-j\omega t}$

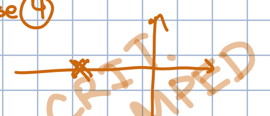
case ③ UNDAMPED



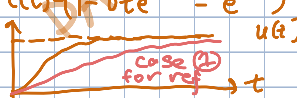
$c(t) = 1 - e^{j\omega t} = (1 - C_1\cos\omega t)u(t)$



case ④ CRIT. DAMPED



$c(t) = (1 - \sigma t e^{-\sigma t} - e^{-\sigma t})u(t)$



time-domain solution to general form:

$$c(t) = 1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \cos\left(\omega_n\sqrt{1-\xi^2}\, t - \phi\right) \quad \text{where } \phi = \tan^{-1}\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)$$

Value	Expression
Rise Time	$T_r = \frac{\pi - \phi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi-\phi}{\omega_d}$
Settling Time	$T_s \approx 4/\omega_n\xi = 4/\sigma_d$
Peak Time	$T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{\omega_d}$
%OS	$OS = e^{-(\xi\pi/\sqrt{1-\xi^2})} * 100$

new definitions

Aside: having zeros or > 2 poles renders all of these equations obsolete UNLESS
- approximate away smol zeros
- approximate away large poles (Real component >5x away).

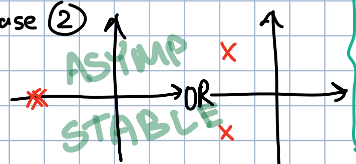
STABILITY

consider the poles of an LTI system $\lambda_1, \lambda_2, \ldots$

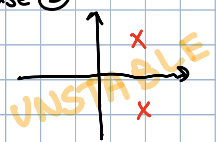
$y_N(t) = \Sigma\, \alpha_i e^{\lambda_i t}$

case (1)

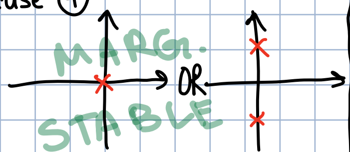
case (2)



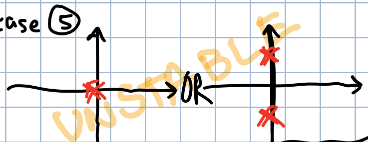
Case (3)



case (4)



case (5)



Solving Time-Response of systems analytically

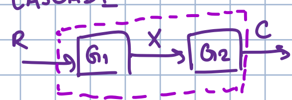
if $n = 1$: $\dot{x} = ax + bu$, $y = x$ $\Rightarrow y(t) = x(0)e^{at} + \int_0^t e^{a(t-\tau)} b u(\tau) d\tau$

if $n \geq 2$: $\dot{x} = Ax + Bu$, $y = x$ $\Rightarrow y(t) = x(0)e^{At} + \int_0^t e^{A(t-\tau)} Bu(\tau)dt$

//Note, if $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $\underbrace{e^{At}}_{\phi(t)\ \triangleq\ \text{"state-transition matrix"}} = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{bmatrix}$ (Proof is in diagonalization)

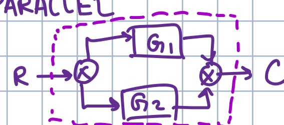
Block Diagram Notation

CASCADE:



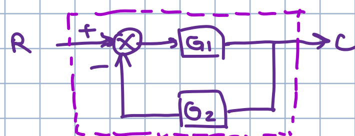
$G_T = G_1 * G_2$

PARALLEL



$G_T = G_1 + G_2$

NEGATIVE FEEDBACK:



where G_1: "feed-forward gain"
G_2: "feed-back gain"

$G_T = \frac{G_1}{1 + G_1 G_2}$ //proof is simple algebra

ROUTH TEST for finding higher-order poles

Ex: $a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s^1 + a_0 s^0$

s^4	a_4	a_2	a_0
s^3	a_3	a_1	$\emptyset$
s^2	b_1	b_2	$\emptyset$
s^1	b_3	$\emptyset$	$\emptyset$
s^0	$\vdots$		

"pivot" for finding the next row.

$b_1 = \frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3}$ $\quad b_2 = \frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & \emptyset \end{vmatrix}}{a_3}$ $\quad b_3 = \frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1}$

→ # of sign changes in the 1st col of Routh Table is # RHP poles

General Routh Algorithm.

1. Consider a 2x2 matrix obtained by the 1st column of the 2 preceding rows & column right above b_{i+1}
2. Calculate the negative determinant
3. Divide by 1st element of preceding row.

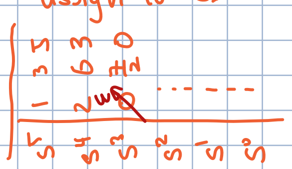
ROUTH EXCEPTIONAL CASES

(1) Row that begins with a zero

Algorithm: - Replace the 0 with a dummy variable like ε
- continue solving as if ε is any number

Caveat: solution DOES NOT depend on what value / sign you assign to ε.

Ex:



Regardless of $\varepsilon = \pm 10^{-6}$, there will be 2 sign changes

(2) Entire Row of Zeros. (poles might exist on jω-axis)

$\vdots$			
s^k	e_k	e_{k-1}	$\ldots$
s^{k-1}	0	0	0 ...
$\vdots$			

↳ row of all zeros.

Step (1) ★ important ★
Form $P(s) = e_k s^k + e_{k-1} s^{k-2} + \ldots + e_{k-2} s^{k-4}$
Original Polynomial = $P(s) * Q(s)$
# of sign changes from row 0 → k is about the poles of Q(s)

Step (2) Take derivative of P(s) = P'(s)

Step (3) Replace row of zeros with coefficients for P'(s)

$\boxed{0, 0 \ldots 0} \Rightarrow \boxed{ke_k, (k-2)e_{k-1}, \ldots}$

Step (4) solve for poles of P(s)

jω poles = $\underset{\text{(order of P)}}{K}$ − 2 * # RHP poles of P(s)

Steady-State Error

$e(t) \triangleq r(t) - c(t)$ $\min\left(\lim_{t\to\infty}(e(t))\right)$

$e(\infty) = 0$ "perfect tracking"

$= \text{finite}$ "tracking w/ finite error"

$= \infty$ "fail"

Final Value Thm for finding SS error:

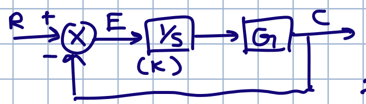
$e(\infty) = \lim_{s\to 0} sE(s) = s(R(s) - C(s)) = s(R(s) - G(s)R(s))$

General Solution to achieving perfect tracking is to add n integrators as your controller.

TABLE:

Input	Controller	SS-error (NO controller!)
$1/s$ (step)	$1/s$	$\frac{1}{1+\lim_{s\to 0} G(s)}$ → K_p position const.
$1/s^2$ (ramp)	$1/s^2$	$\frac{1}{\lim_{s\to 0} sG(s)}$ → K_v veloc. const.
$1/s^3$ (parab.)	$1/s^3$	$\frac{1}{\lim_{s\to 0} s^2 G(s)}$ → K_a acl. const.

Example of Integrator Controller on Step-input.



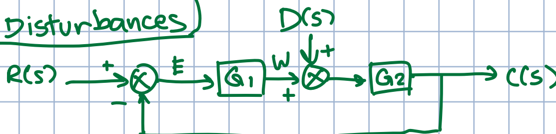
$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s}{s+G(s)} = 0$

As long as $G(s) \neq 0$

Q: What if $G(0) = 0$? AKA zero at the origin.

A: slap n EXTRA integrators where n = # of zeros at origin of $G(s)$

Disturbances



$$\boxed{C(s) = \frac{G_2}{1+G_1G_2} D + \frac{G_1G_2}{1+G_1G_2} R}$$

//general form of controller w/ disturbance

$E(s) = R(s) - C(s) = \frac{-G_2}{1+G_1G_2} D + \frac{1}{1+G_1G_2} R$

under stability prerequisite

$e(\infty) = e_R(\infty) + e_D(\infty)$

$e_R(\infty) = \lim_{s\to 0} \frac{s}{1+G_1G_2} R(s)$

$$\boxed{e_D(\infty) = \lim_{s\to 0} \frac{-G_2 s}{1+G_1G_2} D(s)}$$

SS-error using State-space.

$R(s) \to \boxed{\text{Some CL system}} \to C(s)$

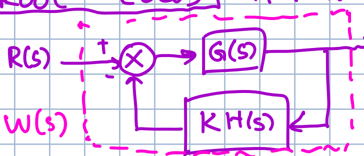
$C(s) = [C(sI-A)^{-1}B + D] R(s)$

$E(s) = [1 - (C(sI-A)^{-1}B + D)] R(s)$

$$\boxed{e(\infty) = \lim_{s\to 0} sR(s)[1 - (C(sI-A)^{-1}B + D)]}$$

Root-Locus

A feed-forward gain does NOT change poles



Assuming $H(s)$ & $G(s)$ are both rational functions,

$$\boxed{W(s) = \frac{G}{1+KGH}}$$

//Tuning K changes poles

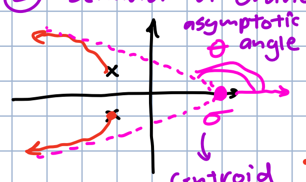
Poles occur when $1 + KGH = 0$, → $GH = -\frac{1}{K}$ → $|GH| = \frac{1}{K}$ → as $K\to 0$ $|GH| = \infty$; as $K\to\infty$ $|GH| = 0$.

Rules of RL

① let GH be a proper TF with deg(num) = m & deg(denom) = n & $m \le n$

- When $K=0$, RL trajectory starts at the poles
- As $K\to\infty$, RL trajectories approach the zeros
- $n-m$ trajectories go to ∞

② Behavior of branches going towards ∞

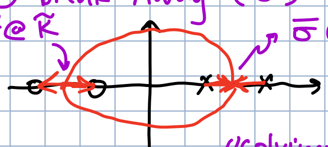


$$\boxed{\theta = \frac{(2r+1)180}{n-m} \qquad \sigma = \left(\sum_{j=1}^{n} P_j - \sum_{i=1}^{m} Z_i\right)/n-m}$$

$r = 0, \pm 1, \pm 2, \ldots$

③ Break-Away ($\bar{\sigma}$) & Break-In ($\tilde{\sigma}$) pts

$\tilde{\sigma} @ \tilde{K}$ $\bar{\sigma} @ \bar{K}$



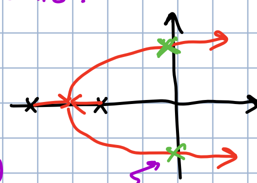
$$\boxed{\sum_{i=1}^{m} \frac{1}{\sigma - z_i} = \sum_{j=1}^{n} \frac{1}{\sigma - P_j}}$$

//solving this equation yields all breakaway & break-in points

④ Where do we hit the $j\omega$ axis?

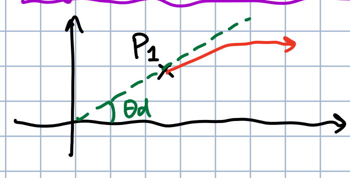
Algorithm:

1. Form Routh Table in terms of K
2. Solve Routh Table
3. Find first row where properly setting K gives full row of zeros
4. Find $\hat{K}$ at which row is zero & solve for poles



$@\hat{K} = x$, $\hat{s} = \pm j\omega$

⑤ Angle of arrival and departure.

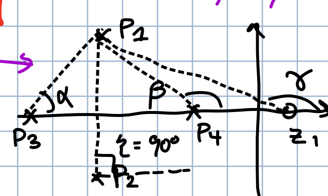


$$\boxed{\theta_d = \sum_i \measuredangle(P_1 Z_i) - \sum_j \measuredangle(P_1 P_j) + 180(2r+1)}$$

where $r = 0, \pm 1, \pm 2, \ldots$

concrete example

$$\boxed{\theta_d = \gamma - (\alpha + \beta + \xi) + 180(2r+1)}$$



Angle of arrival is the exact same, just replace P_1 with Z_1

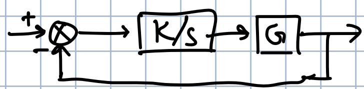
Practical Design via RL

Higher-Order → 2nd-order apprx.

Criteria: ① P_1 & P_2 are dominant. (5x rule)

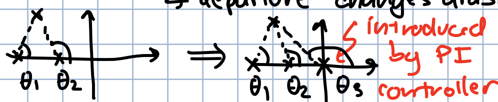
② every dominant zero should be NEAR a dominant pole to cancel out.

I-controller



$e(\infty) \to 0$

Problem with this is the introduction of a pole w/o an accompanying zero means ∡ departure changes drastically.



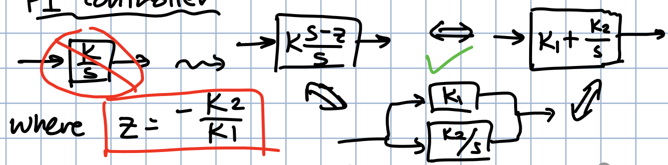
Strategy

① use desired T_r, T_s, T_p, %OS to find the location of the desired poles. (or possible desired poles)

② Tune RL gain K to achieve target.

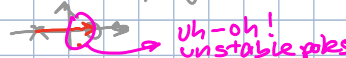
③ IF there is more than 1 possible pair, use these rules to find which is most 2nd-order

Solution → PI controller

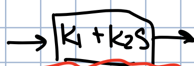


where $\boxed{z = -\frac{K_2}{K_1}}$

NOTE: introduced zero MUST be in the LHP, otherwise at certain values of K, system becomes UNSTABLE



PD controller → useful for pole tuning



$\boxed{z = -K_1/K_2}$

useful for when P_1' & P_2' (desired poles) are NOT on the current RL trajectory.

Intuition: Add a LHP zero to the system

Design Strategy: ① Identify desired P_1', P_2'

② Find it's ∡ to the origin

③ use ∡ departure formula to find where to place θ_2 such that.

$$\boxed{\theta_2 + \sum_i \measuredangle(P_1' z_i) - \sum_j \measuredangle(P_1' z_j) = 180(2r+1)}$$

LEAD compensator → passive analog to PD

where $|p'| > |z_1'|$

in practice, differentiators do not handle noise well so lead compensators are effectively integrators

$$\frac{s - z_1'}{s - p'} = 1 + \frac{p' - z_1'}{s - p'}$$ → behaves like an integrator

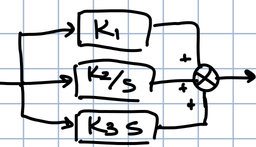
LAG Compensator (passive implementation of PI)

Active Control requires Op Amps for $p = 0$

Passive control leverages passive circuits

Design guideline: $\boxed{\text{as } \frac{z}{p} \to \infty,\ e_{ss}(\infty) \to 0}$

PID controllers



PID = PI + PD } 2 pole, 1 zero

PI: 1 pole, 1 zero; PD: 1 zero

$$\boxed{K_1 + K_2/s + K_3 s = \frac{K_3\left(s^2 + \frac{K_1}{K_3}s + \frac{K_2}{K_3}\right)}{s}}$$

general PID guide (not set in stone)

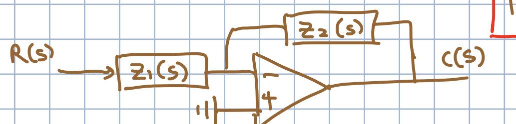
① Design a PD controller to achieve desired behavior

② Augment controller with plant to obtain new system G'

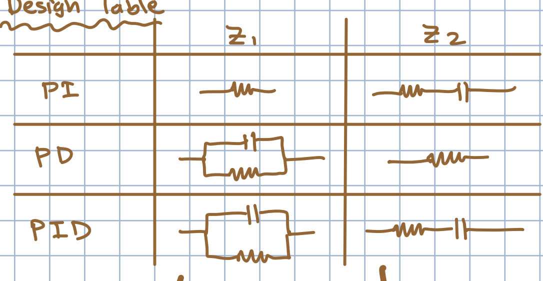
③ design a PI controller on G' to set $e(\infty) \to 0$

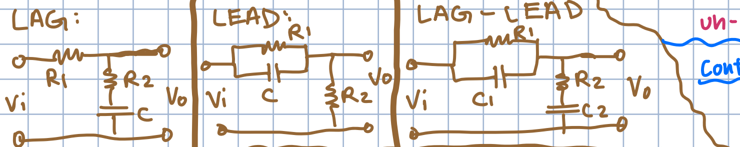
IMPLEMENTATION

$$\boxed{TF = \frac{C(s)}{R(s)} = -\frac{Z_2(s)}{Z_1(s)}}$$



Design Table





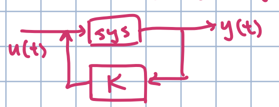
LAG: $\frac{V_o}{V_i} = \frac{1}{\beta}\left(\frac{s + T^{-1}}{s + (\beta T)^{-1}}\right)$, $T = R_2 C$, $\beta = \frac{R_1 + R_2}{R_2}$

LEAD: $\frac{V_o}{V_i} = \frac{\left(s + \frac{1}{T}\right)}{\left[s + \frac{1}{\alpha T}\right]}$, $T = R_1 C$, $\alpha = \frac{R_2}{R_1 + R_2}$

LAG-LEAD: $\frac{V_o}{V_i} = \frac{(s + T_1^{-1})(s + T_2^{-1})}{(s + \beta T_1^{-1})(s + \beta T_2^{-1})}$, $T_1 = R_1 C_1$, $T_2 = R_2 C_2$, $\beta = \frac{R_1 + R_2}{R_2}$

MODERN CONTROLS

SISO vs MIMO systems



where $K \in m \times n$ matrix.

if $u(t) \in \mathbb{R}^m$, $y(t) = x(t) \in \mathbb{R}^n$ → $K \in \mathbb{R}^{m \times n}$

system: $\dot{x} = Ax + Bu$

$u = Kx$

$\dot{x} = \boxed{(A + BK)}x$ ← new effective A-matrix

Tuning the K-matrix so (A+BK) has the desired eigenvals $(\lambda_1, \dots, \lambda_n)$ ⟺ poles

Ex: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $K = [K_1\ K_2]$

$\det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \left(\begin{bmatrix} K_1 & K_2 + 1 \\ K_1 & K_2 \end{bmatrix}\right)\right) = 0$

$(s - K_1)(s - K_2) - K_1(K_2 + 1) = 0$

uh-oh: nonlinear term

$K_1 K_2$ is hard to solve for, how to make it easier? segway,

CHANGE STATES (z)

use $\boxed{T_{n \times n}}$ (transformation matrix)

$\boxed{z = Tx}$ $T: \mathbb{R}^n \leftrightarrow \mathbb{R}^n$

$\boxed{x = T^{-1} z}$

$\dot{x} = Ax + Bu$

$u = Cx$

$$\boxed{\begin{aligned} \dot{z} &= TAT^{-1}z + TBu \\ u &= CT^{-1}z \\ \tilde{A} &= TAT^{-1} \quad \tilde{B} = TB \\ \tilde{C} &= CT^{-1} \end{aligned}}$$

Controllability

x_0 → arbitrary init. state

x_f → arbitrary final state

t_f → arbitrary ref. time.

"controllable" iff for $\dot{\hat{x}} = A\hat{x} + B\hat{u}$ $\exists\ \hat{u}$ st. $\hat{x}(0) = x_0 \implies \hat{x}(t_f) = x_f$

controllable iff

$\boxed{[B\ \ AB\ \ A^2B \dots A^{n-1}B] \text{ is full-rank (Rows are lin. indep.)}}$

→ "controllability Matrix"

Controllable Canonical Form (avoids non-linearities when attempting to coeff-match when pole-placing)

Steps|

① Convert system to controllable-canonical (A', B')

② Design a controller for NEW model K'

③ Bring controller back to original representation K

$$\begin{cases} x = Pz & \text{where } P \text{ is to be designed} \\ \dot{x} = Ax + BKu \end{cases} \Rightarrow \boxed{\dot{z} = \underbrace{P^{-1}AP}_{A'}z + \underbrace{P^{-1}B}_{B'}\underbrace{KP}_{K'}u}$$

$$A' = \begin{bmatrix} 0 & & \\ \vdots & & I \\ 0 & & \\ -a_0 & \dots & -a_{n-1} \end{bmatrix} \quad B' = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \Big\} \text{controllable-canonical form}$$

// in this form, solving $\det(sI - (A' + B'K'))$ is ensured to have coefficients linear in $k_1 \dots k_n$, EASY WORK ☺

let $\boxed{C_x = [B \;\; AB \;\dots\; A^{n-1}B]}$
$\boxed{C_z = [B' \;\; A'B' \;\dots\; A'^{n-1}B']}$ $\boxed{C_z = P^{-1}C_x}$ $\boxed{P = C_x C_z^{-1}}$

//this is how you find P

//NOTE: $A + BK$ & $A' + B'K'$ have the same eigenvalues

$\boxed{\text{Find } K'}$ s.t. $A' + B'K'$ has desired eigenvals $(\lambda_1 \dots \lambda_n)$

$\boxed{\text{backtrack to } K}$ $\boxed{K = K'P^{-1}}$

OBSERVABILITY essence: can we backtrack?

$\begin{matrix} \dot{x} = Ax \\ y = Cx \end{matrix}\Big\}$ without knowing x, but with observation of y, can we find $x(t)$ for all $t \in [0, T]$ regardless of initial state x_0.

Observability Matrix.

$$\boxed{\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \text{ observable iff } \underline{O} \text{ is full-rank}}$$

Designing L STEP ① stabilizing error dynamics

$$\begin{matrix} \dot{x} = Ax + Bu \\ y = Cx \end{matrix}\Big\} \begin{matrix}\text{Actual} \\ (x_0\ ???)\end{matrix} \qquad \begin{matrix} \dot{\hat{x}} = A\hat{x} + B\hat{u} \\ \hat{y} = C\hat{x} \end{matrix}\Big\} \begin{matrix}\text{simulated} \\ (\hat{x}_0 = 0)\end{matrix}$$

state error: $e(t) = x - \hat{x}$

$$\xrightarrow{\text{Algebra}} \dot{e}(t) = A(x - \hat{x}) + Bu - Bu - L(y - \hat{y})$$
$$= Ae(t) + 0 - LC(x - \hat{x})$$
$$\boxed{\dot{e}(t) = (A - LC)e(t)}$$

// Design L such that $A - LC$ is stable $\hookrightarrow e(\infty) \to 0$

STEP ③ Find $\tilde{A} - \tilde{L}\tilde{C}$

Coefficient match $\det(sI - (\tilde{A} - \tilde{L}\tilde{C}))$ with DESIRED governing polynomial

STEP ② Observable Canonical Form

$$\tilde{A} = \begin{bmatrix} -a_{n-1} & & \\ \vdots & & I \\ -a_1 & & \\ -a_0 & 0 & \dots & 0 \end{bmatrix}; \quad \tilde{C} = [1\ 0 \dots 0]$$

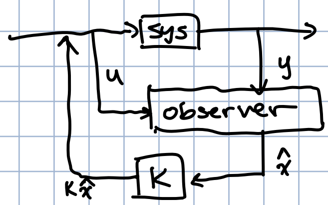
Find observability matrix of original system

$$\underline{O_x} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Find observability matrix of canonical system. using coeffs of $\det(sI - A)$

$$\underline{O_z} = \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \vdots \\ \tilde{C}\tilde{A}^{n-1} \end{bmatrix}$$

Controller + Observer Combo



Controller design

$P = C_x C_z^{-1}$

transformation; controllability matrix; coeff. matrix in canonical form

$K = K'P^{-1}$

controller for original sys; controller for canonical system

Observer design

$P = O_x^{-1} O_z$

transformation; observability matrix; obs. matrix in canonical form

$L = P\tilde{L}$

observer original; observer canonical

Objective: Design K & L to make $\underset{\text{stabilize the output}}{A + BK}$ & $\underset{\text{stabilize the error.}}{A - LC}$ both stable.

Separation Theorem: Proof K & L can be designed independently

$$① \begin{cases} \dot{x} = Ax + Bu \\ u = K\hat{x} \end{cases} \qquad ② \begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \\ e = x - \hat{x} \end{cases}$$

① & ② : $\dot{x} = Ax + BK(x - e)$
$= (A + BK)x - BKe$ ⊛

Also Recall ⊛⊛
$\dot{e} = (A - LC)e$

$$\boxed{\begin{matrix}⊛ \\ ⊛⊛\end{matrix} \begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \underbrace{\begin{bmatrix} A + BK & -BK \\ 0 & A - LC \end{bmatrix}}_{\bar{A}} \begin{bmatrix} x \\ e \end{bmatrix}}$$

The eigenvals of $\bar{A}$
= poles of entire system with K & L implemented
= eigs$(A + BK) \cup$ eigs$(A - LC)$

LQR / LQG

Consider $\dot{x} = Ax + Bu$

LQR : $\boxed{\min_{x(t)\, u(t)} \int_0^\infty x^TQx + u^TRu \, dt}$

where $Q: \mathbb{R}^{n \times n}$ $\quad R: \mathbb{R}^{m \times m}$

Assume $n = 1$

LQR becomes $\underset{\text{(state regulation)}}{q\int_0^\infty x^2 dt} + \underset{\text{(input-energy minimization)}}{r\int_0^\infty u^2 dt}$

L: linear system
Q: quadratic in x & u
R: regulates states to zero

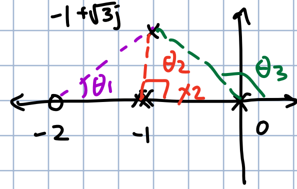
WHAT IS OPTIMAL K? given A, B, R, Q

$\boxed{K = R^{-1}B^TP}$ //system of 2 matrices K & P

where $\boxed{A^TP + PA - PBR^{-1}B^TP + Q = 0}$

Practice Final Question 3

a) $G(s) = \frac{s+2}{s(s+1)^2}$ want poles @ $-1 \pm \sqrt{3}j$



current poles: -1 x2 & 0
current zeros: -2

Does $\theta_1 - 2 * \theta_2 - \theta_3 = 180(2r+1)$
$60° - 90° \times 2 - 120°$
$= -240° \neq 180(2r+1)$

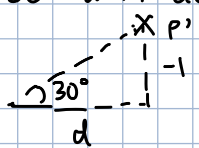
$\boxed{\text{currently no } K_p \text{ exists to place the poles @ } -1 \pm \sqrt{3}j}$

b) angle of deficiency $\triangleq$ "additional angle needed"
$|180° - 240°| = \boxed{60°}$

c) Lead compensator design
(needs to satisfy angle of deficiency)

$C(s) = K \frac{s+1}{s+\alpha}$ → already introduced zero at -1 (90°)

new $\Sigma\theta = -240° + 90° = -150°$
need to place α s.t. pole makes 30° with desired pole.



$d = 3$
thus $\boxed{\alpha = -4}$

which value of K yields P' ?
$K = \frac{\Pi \text{ distance to all poles}}{\Pi \text{ distance to all zeros}} = \boxed{6}$

d) $K_v = \lim_{s \to 0} s\,G(s)$

$K_v = \lim_{s \to 0} \cancel{s} \frac{s+2}{\cancel{s}(s+1)^2} K \frac{s+1}{s+\alpha} = \left(\frac{2}{1}\right)\left(\frac{1}{4}\right) 6 = \boxed{3}$

e) Lag compensator just needs to adjust for K_v
so ss-gain needs to be 10,
can choose $\frac{s+0.1}{s+0.01}$

full lag-lead : $\boxed{6 \frac{s+1}{s+4} \frac{s+0.1}{s+0.01}}$

Review Question 5

Review Short Answer Q2.
Controllability & Observability do not depend on which state is being controlled or observed,
AKA $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $C = [1 0 0]$ $[0 1 0]$ or $[0 0 1]$
// proved by evaluating 3x3 $\underline{\mathcal{C}}$ & $\underline{\mathcal{O}}$ matrices which is too much damn work.